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Fixed Point Theorems in Banach Spaces Over Topological Semifields

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Abstract. A fixed point theorem for five pairwise commuting mappings on a Banach space X over a topological semifield is proved which improves and extends the main result of [5]. In the sequel, an application is given for the solvability of certain non-linear functional equation in X.

1. Introduction

Autonovskii, Bolyouskii and Saymsakov [1] have shown that the class of real Hausdorff locally convex spaces coincides with the class of linear spaces, normed over some topological semifields.

Let E be a topological semifield and K be the set of all its positive elements. For any two elements x, y in E, if $y - x$ is in $\bar{K}$ (in K) this is denoted by $x << y$ ($x < y$). In [1], it has been proved that every topological semifield E contains a subsemifield, called the axis of E, which is isomorphic to the field $\mathbb{R}$ of real numbers. As a consequence, by identifying the axis and $\mathbb{R}$, each topological semifield can be regarded as a topological linear space over the field $\mathbb{R}$.

If there exists a mapping $d : X \times X \to E$ satisfying the usual axioms for a metric (see [1], [2], and [4]), then the ordered triple (X, d, E) is said to be a metric space over the topological semifield E.

In this paper, we consider linear spaces defined on the field $\mathbb{R}$. Let X be a linear space. The ordered triple $(X, \|\cdot\|, E)$ is called a *feeble normed space over the topological semifield* E if there exists a mapping $\|\cdot\| : X \to E$ satisfying the usual axioms for a norm (see [1] and [3]).

We use the following definition in our main result:

Definition 1. Let $(X, \|\cdot\|, E)$ be a feeble normed space over a topological semifield E, and let $d(x, y) = \|x - y\|$ for all x, y in X. A space $(X, \|\cdot\|, E)$ is said to be a Banach space over the topological semifield E if (X, d, E) is a sequentially complete metric space over the topological semifield E.

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2. MAIN RESULT

Theorem 1. *Let X be a Banach space over a topological semifield E, and let F, G, H, A and B be five continuous self-mappings of X satisfying the following conditions:*

(1) $G(X) \supset (1-t)G(X) + tFA(X) : \forall t \in (0,1]$,
(2) $G(X) \supset (1-t)G(X) + tHB(X) : \forall t \in (0,1]$,
(3) *the mappings F, G, H, A and B are pairwise commuting,*
(4) $p\|Gx - Gy\|^m + \|Gy - HBy\|^m << q\|Gx - FAx\|^m$,
(5) $p\|Gx - Gy\|^m + \|Gy - FAy\|^m << q\|Gx - HBx\|^m$,

for all x, y in X, where $p, m > 0$, and $0 < q < 1$. Then for the sequence $\{Gx_n\}$ defined by

$$Gx_{2n+1} = (1-t_{2n})Gx_{2n} + x_{2n}FAx_{2n}, \tag{6}$$

$$\sum_{n=1}^{\infty} \prod_{\gamma=0}^{n-1} \left(1 - \frac{p}{q} t_\gamma^m\right)^{1/m} \tag{7}$$

is bounded.

Proof. Let x_0 be an arbitrary point in X. From (6) and (7), we have

$$\|Gx_{2n+1} - Gx_{2n}\| = t_{2n}\|FAx_{2n} - Gx_{2n}\|, \tag{8}$$

$$\|Gx_{2n+2} - Gx_{2n+1}\| = t_{2n+1}\|HBx_{2n+1} - Gx_{2n+1}\|. \tag{9}$$

Put $x = x_{2n}$ and $y = x_{2n+1}$ in (4). Then from (8) and (9), we obtain

$$p\|Gx_{2n} - Gx_{2n+1}\|^m + t_{2n+1}^{-m}\|Gx_{2n+2} - Gx_{2n+1}\|^m << qt_{2n}^{-m}\|Gx_{2n+1} - Gx_{2n}\|^m,$$

which implies that

$$\|Gx_{2n+2} - Gx_{2n+1}\| << \frac{t_{2n+1}}{t_{2n}}\left(q - pt_{2n}^m\right)^{1/m}\|Gx_{2n+1} - Gx_{2n}\| \tag{10}$$

for all n.

Putting $x = x_{2n+1}$ and $y = x_{2n+2}$ in (5) and using (8) and (9) we obtain

$$t_{2n+2}^{-m}\Big(pt_{2n+2}^m\|Gx_{2n+1} - Gx_{2n+2}\|^m + \|Gx_{2n+3} - Gx_{2n+2}\|^m\Big) << \\ << qt_{2n+1}^{-m}\|Gx_{2n+2} - Gx_{2n+1}\|^m$$

for all n. Hence,

$$\|Gx_{2n+3} - Gx_{2n+2}\| << \frac{t_{2n+2}}{t_{2n+1}}\left(q - pt_{2n+1}^m\right)^{1/m}\|Gx_{2n+2} - Gx_{2n+1}\| \tag{11}$$

for all n. From (10) and (11) we then obtain

$$\|Gx_n - Gx_{n+1}\| << \frac{t_n}{t_{n-1}}\left(q - pt_{n-1}^m\right)^{1/m}\|Gx_{n-1} - Gx_n\|,$$

which implies that

$$\|Gx_n - Gx_{n+1}\| << \frac{t_n}{t_0} \prod_{\gamma=0}^{n-1} \left(q - pt_\gamma^m\right)^{1/m} \|Gx_0 - Gx_1\|,$$

or equivalently

$$\|Gx_n - Gx_{n+1}\| << \frac{t_n}{t_0} q^{1/m} \prod_{\gamma=0}^{n-1} \left(1 - \frac{p}{q} t_\gamma^m\right)^{1/m} \|Gx_0 - Gx_1\|.$$

Since

$$\sum_{n=1}^{\infty} \prod_{\gamma=0}^{n-1} \left(1 - \frac{p}{q} t_\gamma^m\right)^{1/m}$$

is bounded, it follows that $\{Gx_n\}$ is a Cauchy sequence. Since X is complete, it then follows that the sequence $\{Gx_n\}$ converges to a point u in X. Using (6) and (7), we see that $\{FAx_{2n}\}$ and $\{HBx_{2n+1}\}$ also converge to u. Since F, A, H and B are continuous, we have

$$FA(Gx_{2n}) \to FAu, \quad HB(Gx_{2n+1}) \to HBu. \tag{12}$$

Since G commutes with F, A, H and B, we have

$$FA(Gx_{2n}) = G(FAx_{2n}), \quad HB(Gx_{2n+1}) = G(HBx_{2n+1})$$

for $n = 0, 1, 2, \ldots$ Letting $n \to \infty$, we have

$$FAu = Gu = HBu$$

and then

$$\begin{aligned} G(Gu) = G(FAu)S = FA(Gu) = FA(HBu) \\ = G(HBu) = HB(Gu) = HB(HBu). \end{aligned} \tag{13}$$

Now if $FAu \neq HB(FAu)$, then by (4), (12) and (13) we have

$$p\|Gu - G(HBu)\|^m + \|G(HBu) - HB(HBu)\|^m << q\|Gu - FAu\|^m,$$

and

$$p\|FAu - HB(FAu)\|^m + \|FAu - HB(FAu)\|^m << q\|FAu - FAu\|^m.$$

Hence,

$$FAu = HB(FAu). \tag{14}$$

By (4), (13) and (14), we have

$$FAu = HB(FAu) = G(FAu) = FA(FAu),$$

which implies that FAu is a common fixed point of FA, G and HB.

Now, we shall prove the uniqueness of the common fixed point of FA, G and HB. Suppose that u and v are two common fixed points of FA, G and HB in X. Then by (4), we have

$$p\|Gu - v\|^m + \|Gv - HBv\|^m << q\|Gu - FAu\|^m,$$

which implies

$$p\|u-v\|^m + \|v-v\|^m << q\|u-u\|^m$$

and so

$$\|u-v\|^m << 0.$$

This implies the uniqueness of the common fixed point of FA, G and HB in X.

Now by (3) we have

$$Aw = A(FAw) = FA(Aw), \quad Aw = A(Gw) = G(Aw)$$

and

$$Fw = F(Gw) = G(Fw), \qquad Fw = F(FAw) = FA(Fw),$$

showing that Aw and Fw are common fixed points of FA and G.

Similarly, we could prove that Bw and Hw are common fixed points of HB and G.

By (4), we have

$$\begin{aligned} p\|Aw-Bw\|^m + \|Bw-Bw\|^m &= p\|GAw-GBw\|^m + \|GBw-HBBw\|^m \\ &<< q\|GAw-FAAw\|^m = q\|Aw-Aw\|^m = 0 \end{aligned}$$

and so

$$\|Aw-Bw\|^m << 0.$$

This implies $Aw = Bw$.

Similarly, by (5) we obtain $Fw = Hw$. Since w is the unique common fixed point of the pairs FA, G and HB, G, we obtain

$$Aw = Bw = Fw = Hw = w = FAw = HBw = Gw.$$

This completes the proof of the theorem. □

Remark 1. By setting $A = B = I$ (the identify mapping on X) and $\{t_n\} \equiv \{t\}$ where $0 \leq q - pt^m < 1$, we obtain Theorem 1 of Pathak, Lakshmi, Taş and Fisher [5].

3. An Application

In this section, we investigate the solvability of certain non-linear functional equations in a Banach space over a topological semifield.

Theorem 2. *Let X be a Banach space over a topological semifield E, and let F, G, H, A and B be five continuous self-mappings on X satisfying conditions (1) – (5) of Theorem 1. Let $\{g_{p'}\}$, $\{f_{p'}\}$ and $\{h_{p'}\}$ be sequences of elements in X and let $w_{p'}$ be the unique solution of the system of equations*

$$u - Gu = g_{p'}, \quad Gu - FAu = f_{p'}, \quad Gu - HBu = h_{p'}.$$

Proof. If

$$\lim_{p'\to\infty} \|g_{p'}\| = \lim_{p'\to\infty} \|f_{p'}\| = \lim_{p'\to\infty} \|h_{p'}\| = 0,$$

then the sequence $\{w_{p'}\}$ converges to the solution of the equations

$$u = Gu = Fu = Hu = Au = Bu.$$

Suppose that

$$\|w_{p'} - Gw_{p'}\| \neq 0, \quad \|Gw_{p'} - FAw_{p'}\| \neq 0, \quad \|Gw_{p'} - HBw_{p'}\| \neq 0.$$

Then by (4), we have for $p' > q'$,

$$\begin{aligned} \|w_{p'} - w_{q'}\| &<< \|w_{p'} - Gw_{p'}\| + \|Gw_{p'} - Gw_{q'}\| + \|Gw_{q'} - w_{q'}\| << \\ &<< \|g_{p'}\| + p^{-1}\big[q\|Gw_{p'} - FAw_{p'}\|^m - \\ &\qquad - \|Gw_{q'} - HBw_{q'}\|^m\big]^{1/m} + \|g_{q'}\| = \\ &= \|g_{p'}\| + p^{-1}\big[q\|f_{p'}\|^m - \|h_{q'}\|^m\big]^{1/m} + \|g_{q'}\|. \end{aligned} \tag{15}$$

Letting $p', q' \to \infty$, it follows that $\|w_{p'} - w_{q'}\| \to 0$, which implies that $\{w_{p'}\}$ is a Cauchy sequence in X. Since X is complete, it then follows that the sequence $\{w_{p'}\}$ converges to a point w in X.

Since G, F, H, A and B are continuous, it follows from (15) that

$$\|w - Gw\| = \lim_{p'\to\infty} \|w_{p'} - Gw_{p'}\| = \lim_{p'\to\infty} \|g_{p'}\| = 0,$$

$$\|Gw - FAw\| = \lim_{p'\to\infty} \|Gw_{p'} - FAw_{p'}\| = \lim_{p'\to\infty} \|f_{p'}\| = 0,$$

$$\|Gw - HBw\| = \lim_{p'\to\infty} \|Gw_{p'} - HBw_{p'}\| = \lim_{p'\to\infty} \|h_{p'}\| = 0.$$

This implies that $w = Gw = FAw = HBw$. The uniqueness of w as common fixed point of G, F, H, A and B follows by the same argument as in Theorem 1. This completes the proof of the theorem. □

Remark 2. Theorems 1 and 2 and their proofs may be modified for a Banach space over a real or complex field by replacing the symbol "$<<$" with "$\leq$" throughout the text.

Remark 3. By setting $A = B = I$ (the identity mapping on X) in Theorem 2, we immediately obtain Theorem 2 of Pathak, Lakshmi, Tas and Fisher [5].

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